ELSEVIER
Discrete Mathematics 245 (2002) 203–216
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Constructing $\frac{1}{2}$-arc-transitive graphs of valency 4 and vertex stabilizer $Z_2 \times Z_2$

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Received 13 March 2000; revised 14 November 2000; accepted 18 November 2000

Abstract

An infinite familiy of $\frac{1}{2}$-arc-transitive graphs of valency 4 with alternating cycles of length 4 is given. Besides, an infinite family of $\frac{1}{2}$-arc-transitive graphs of valency 4 with vertex stabilizer isomorphic to $Z_2 \times Z_2$ is constructed.

Keywords: Group action; Action digraph; Cayley graph; Vertex-transitive graph; Edge-transitive graph; $\frac{1}{2}$-arc-transitive graph

1. Introduction

Throughout this paper all graphs and groups are assumed to be finite, and unless stated otherwise, the graphs are undirected. For group-theoretic concepts not defined here we refer the reader to [17].

A graph X is said to be $\frac{1}{2}$-*arc-transitive* if its automorphism group $\mathrm{Aut}\,X$ acts vertex- and edge- but not arc-transitively. More generally, by a $\frac{1}{2}$-*arc-transitive action* of a subgroup $G \leqslant \mathrm{Aut}\,X$ on X we shall mean a vertex- and edge- but not arc-transitive action on X. In this case we shall say that X is $(G, \frac{1}{2})$-*transitive*. By a classical result of Tutte [15, 7.53, p. 59], the graph X is of even valency. There has recently been an increased interest in the study of $\frac{1}{2}$-arc-transitive graphs (see [1–8,10–14,16,18]). Among them, graphs of valency 4, the smallest admissible valency, deserve special attention. In [1], an infinite family of such graphs was constructed. Furthermore, the structural properties of tetravalent graphs admitting a $\frac{1}{2}$-arc-transitive group action were studied in [10,11]. In particular, let X be such a graph and $G \leqslant \mathrm{Aut}\,X$, the appropriate group. Then G induces an orientation of the edges of X which is preserved by G. An even length cycle in X is said to be a G-*alternating cycle* if any two of its consecutive

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[1] Supported in part by "Ministrstvo za znanost in tehnologijo Slovenije", proj. no. J1-0496-0101-99.

PII: S0012-365X(01)00142-X

edges have opposite orientation. It was proved in [10] that all G-alternating cycles of X have the same length—half of which is called the *G-radius* of X—and that they give rise to a decomposition of the edge-set of X. (We remark that the group G is omitted as a parameter in the above definitions in case $G = \mathrm{Aut}\,X$.)

In all of the examples of $\frac{1}{2}$-arc-transitive graphs of valency 4 constructed thus far [1,8,10,12], the vertex stabilizer is isomorphic to Z_2, and moreover, the radius is greater than 2. It is the main aim of this paper to construct an infinite family of $\frac{1}{2}$-arc-transitive graphs of valency 4 having vertex stabilizer isomorphic to $Z_2 \times Z_2$. To this end an (auxiliary) infinite family of graphs with radius 2 is first constructed. Before stating these results an additional concept is needed. Let X be a graph of valency 4 admitting a $\frac{1}{2}$-arc-transitive action of a subgroup $G \leqslant \mathrm{Aut}\,X$. We define the graph $\mathrm{Alt}(X)$ as the intersection graph of X with respect to the G-alternating cycles in X. If the G-radius of X is 2 then $\mathrm{Alt}(X)$ has valency 4 and, as it can easily be seen, it admits a $\frac{1}{2}$-arc-transitive action of G with the stabilizer having twice as many elements as that of G on X.

The following two theorems are the main results of this paper. (For convenience we take S_n and A_n to be the group of all permutations and the group of all even permutations, respectively, on the set $\wp_n = \{0, 1, \ldots, n-1\}$ of n letters. Also, note that for a group G and a generating set Q of G such that $1 \notin Q = Q^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, Q)$ *of* G *relative to* Q has vertex set G and edges of the form $[g, gq], g \in G, q \in Q$.)

Theorem 1.1. *Let* $n = 2k+1 \geqslant 17$ *and let* $a = (0\,1 \ldots 2k) \in A_n$ *and* $b = tat^{-1}$, *where* $t = (0\,2)(4\,7) \in A_n$. *Then* $\langle a, b\rangle = A_n$ *and the corresponding Cayley graph* $X_n = \mathrm{Cay}(A_n, Q_{a,b})$, *where* $Q_{a,b} = \{a, b, a^{-1}, b^{-1}\}$, *is* $\frac{1}{2}$*-arc-transitive with valency* 4 *and radius* 2 (*and its automorphism group is isomorphic to* $A_n \times Z_2$).

Theorem 1.2. *With the notation and assumptions of Theorem* 1.1 *the graph* $Y_n = \mathrm{Alt}(X_n)$ *is a* $\frac{1}{2}$*-arc-transitive graph of valency* 4 *with vertex stabilizer isomorphic to* $Z_2 \times Z_2$.

Some combinatorial concepts are given in Section 2 and then used in Section 3 to study the cycle structure of the graphs X_n via certain relations of length at most n in A_n. In particular, a characterization of such relations is obtained. This result is then used in Section 4 to prove Theorem 1.1 and in Section 5 to prove Theorem 1.2.

2. Preliminaries

Let X be a digraph and $v \in V(X)$ a vertex. An arc of X is *incident* with v if v is either its head or its tail. A vertex u is *adjacent* to v (a *neighbor* of v) if either (u, v) or (v, u) is an arc of X. We say that X is *bipartite* if there exists a partition of its vertex set into two subsets such that every arc of X is incident with one vertex in the first subset and one vertex in the second subset. A *walk* in X is an alternating sequence of vertices and arcs $v_0a_0v_1a_1,\dots,v_{l-1}a_{l-1}v_l$ such that for each i we have that a_i is incident with both v_i and v_{i+1}. A walk is *closed* if $v_0=v_l$. A *path* in X is a walk, all of whose vertices are distinct. Let W be a walk in X. Then $|W|$ denotes the length of W, that is, the number of arcs of W. For vertices $u,v\in V(X)$ we let $d(u,v)$ denote the length of the shortest walk from u to v. We say that W *traverses* a vertex v of X if there exists a subpath of length 2 in W with v as its internal vertex.

Let a group G act on the right on a set V (where the action is denoted by $(v,g)\mapsto v*g$), and let Q be a nonempty subset of G. We define the *action* (*multi*)*digraph* $\mathrm{Act}(G,V,Q)$ to be the digraph with vertex set V and arcs of the form $(v,v*q)$, $v\in V$, $q\in Q$. (Note that if $v*q_1=v*q_2$ for $q_1\neq q_2$, then the arcs $(v,v*q_1)$ and $(v,v*q_2)$ are considered to be distinct.) Throughout this paper we shall be assuming that the action of G is transitive and that Q is a generating set of G, thus forcing $\mathrm{Act}(G,V,Q)$ to be (weakly) connected. In particular, if G acts on itself by right multiplication and if $1\notin Q=Q^{-1}$, then the graph associated with the digraph $\mathrm{Act}(G,G,Q)$ is nothing but the Cayley graph, $\mathrm{Cay}(G,Q)$.

For a subset $Q\subseteq G$ we let $\mathrm{Aut}(G,Q)=\{\alpha\in\mathrm{Aut}(G)\colon \alpha(Q)=Q\}$. Next, by a *Q-sequence* and a *Q-relation* we shall mean a word on symbols from $Q\cup Q^{-1}$ which corresponds, respectively, to a simple path and to a simple cycle in $\mathrm{Cay}(G,Q\cup Q^{-1})$. (In other words, by a Q-relation we mean a primitive Q-relation and by a Q-sequence a reduced word on symbols from $Q\cup Q^{-1}$, that is, a word such that no proper subword is a relation.) We say that two Q-sequences are *equivalent* if one can be obtained from the other by a finite series of transformations of the following three types: a cyclic rotation, taking the Q-sequence in the reverse order with all terms inverted (that is, the inverse Q-sequence), or substituting each term in the Q-sequence by its image under a fixed element of $\mathrm{Aut}(G,Q\cup Q^{-1})$. Note that the corresponding equivalence relation on Q-sequences distinguishes between relations and nonrelations in G. To each Q-sequence in a group G acting on a set V and a vertex v of $\mathrm{Act}(G,V,Q)$, we may associate in a natural way a walk originating in v. Furthermore, if the action of G on V is faithful, then a Q-sequence in G is a relation if and only if it represents a closed walk at every vertex of $\mathrm{Act}(G,V,Q)$. In this sense the action digraph is a useful geometric tool for testing whether a given sequence is a group relation or not.

For a Q-sequence S in G let $l(S)$ denote the *length* of S. In particular, if $Q\cap Q^{-1}=\emptyset$, we may further define the *positive length* $l^+(S)$ and the *negative length* $l^-(S)$ of S, respectively, as the numbers of terms of S belonging to Q and to Q^{-1}. Note that the functions l^+ and l^- are constant on each equivalence class of Q-sequences. We say that a Q-sequence S is *balanced* if $l^+(S)=l^-(S)$.

3. Relations in A_n

Recall that in the statement of Theorem 1.1 we have introduced the following notation, which will be used throughout the rest of the paper. We have firstly, $n=2k+1\geqslant 17$; secondly, $a=(0\,1\ldots 2k)\in A_n$ and $b=tat^{-1}$, where $t=(0\,2)(4\,7)$ and

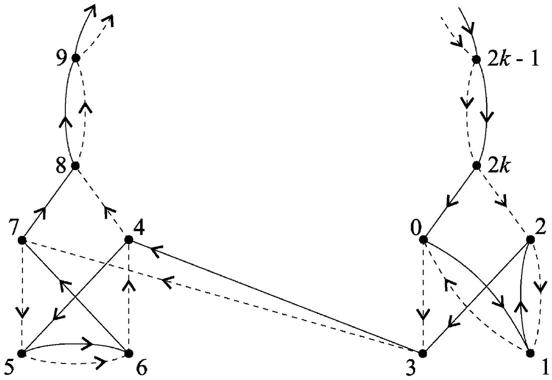

Fig. 1. The action digraph.

thirdly, $Q_{a,b} = \{a, b, a^{-1}, b^{-1}\}$. In order to prove Theorems 1.1 and 1.2 some rather detailed information on the cycle structure of the graphs $X_n = \mathrm{Cay}(A_n, Q_{a,b})$, that is, on $Q_{a,b}$-relations in the group $A_n = \langle a, b\rangle$, is needed. This is the purpose of this section. Hereafter, by a sequence and a relation in A_n we shall always mean a $Q_{a,b}$-sequence and a $Q_{a,b}$-relation in A_n, respectively. For a $Q_{a,b}$-sequence $S = x_1x_2 \ldots x_l$ we let the *code* $c(S)$ of S be the sequence $c(x_1)c(x_2)\ldots c(x_l)$, where for each i we set $c(x_i) = 1$ if $x_i \in \{a, b\}$ and $c(x_i) = 0$ otherwise.

Relations in A_n of length at most n are of particular importance. The analysis of such relations will be done implicitly by considering the associated closed walks in the corresponding action digraph $\Omega_n = \mathrm{Act}(A_n, \wp_n, \{a, b\})$ (see Fig. 1).

We first make a few working definitions about the digraph Ω_n. The subdigraphs of Ω_n induced by the subsets of vertices $\{0, 1, 2, 3, 4, 5, 6, 7\}$ and $\{8, 9, \ldots, 2k\}$ will be called the *hank* H_n and the *bight* B_n, respectively. More precisely, the *right hank* is induced by the set $\{0, 1, 2, 3\}$ and the *left hank* is induced by the set $\{4, 5, 6, 7\}$. Note that all arcs between the two parts of H_n have a tail in the right hank and the head in the left hank. An *attachment arc* is an arc joining a vertex in B_n to a vertex in H_n. We say that a closed walk W in Ω_n is *bighty* if it traverses every vertex of the bight. For a $Q_{a,b}$-sequence S and a vertex v of Ω_n we let $W(S; v)$ denote the walk in Ω_n associated with S whose origin is v . Moreover, the vertex of $W(S; v)$ reached from v by the first r steps will be denoted by $v *_r S$. If $r = l(S)$, then the subscript r is omitted.

The straightforward proof of the next lemma is omitted.

Lemma 3.1. *The digraph obtained from Ω_n by deleting both arcs $(5, 6) = (5, 5 * a) = (5, 5 * b)$ is bipartite.*

From now on we shall refer to each of the two arcs $(5, 6)$ as *singular*. We first take care of relations in A_n having small length.

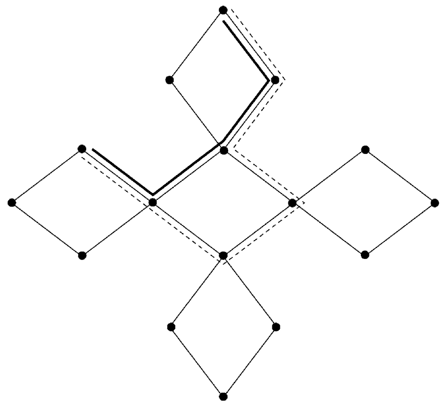

Fig. 2. An expansion and a contraction operation.

Proposition 3.2. *Let $n=2k+1 \geqslant 17$ and let S be a $Q_{a,b}$-relation in A_n. Then $l(S) \geqslant 4$ and moreover, $l(S)=4$ if and only if S is equivalent to the relation $(ab^{-1})^2$.*

Proof. It is clear that there are no relations of length less than 4. As for relations of length 4, they must be balanced. This is easily seen by observing that the corresponding closed walk $W(S;k+4)$ is entirely contained on the bight. Consequently, such a relation is, up to equivalence, either $(ab^{-1})^2$ or a^2b^{-2} or $aba^{-1}b^{-1}$. The first one is indeed a relation for $ab^{-1}=(0\,2)(1\,2k)(3\,6)(4\,7)$, while for the remaining two we have $0 * S=2k$. □

An *expansion* of a $Q_{a,b}$-sequence S in A_n consists of replacing some term x of S, both of whose neighbors are in $\{a,b\}$, by $ba^{-1}b$ if $x=a$ and by $ab^{-1}a$ if $x=b$, or consists of replacing some term x of S, both of whose neighbors are in $\{a^{-1},b^{-1}\}$, by $b^{-1}ab^{-1}$ if $x=a^{-1}$ and by $a^{-1}ba^{-1}$ if $x=b^{-1}$. (Note that the first and the last term of S are also considered as neighbors.) The reverse operation is called a *contraction* (see Fig. 2). Of course, expansions and contractions preserve relations.

The next proposition deals with relations of length strictly greater than 4 and smaller than or equal to n.

Proposition 3.3. *Let $n=2k+1 \geqslant 17$ and let S be a $Q_{a,b}$-relation in A_n of length $4 < l(S) \leqslant n$. Then the following statements hold*:

(i) *If $l(S)$ is odd, then $l(S)=n$ and S is equivalent to the relation a^n.*

(ii) *If $l(S)$ is even, then either S is balanced, or S is not balanced with $n \equiv 2 \pmod 3$ and $l(S) \in \{n-5, n-3, n-1\}$. Moreover, in the unbalanced case, S is equivalent to the relation $S'=(ab)^{(n-5)/2}$ or to a relation obtained from S' by one or two expansions.*

Proof. Let S be a relation of length $4 < l(S) \leqslant n$. Consider the closed walk $W = W(S; k+4)$. We are going to distinguish two cases.

Case 1: *The walk W is not bighty.*

We show that the relation S must be balanced (and hence of even length). It is clear that S is balanced if W is contained entirely on the bight. Hence, we may assume that W has nonempty intersection with the hank. There are two subcases to be considered.

Subcase 1.1: *The walk W contains the segment $[k+4, 2k]$ of B_n.*

Let R be the minimal initial subsequence of S such that $(k+4) * R = 2k$ and let T be the minimal terminal subsequence of S such that $2k * T = k+4$. Then S is of the form $S = RUT$, where $2k * U = 2k$. Clearly, we have $l^+(R) - l^-(R) \geqslant (n-9)/2$ and $l^-(T) - l^+(T) \geqslant (n-9)/2$ and the sequence RT is of course balanced. Since $l(S) \leqslant n$, it follows that $l(U) \leqslant 9$. Moreover, since U has nonempty intersection with the hank, we have $l(U) \geqslant 4$. Consequently, $l^-(R), l^+(T) \leqslant 2$. Now let us consider the closed walk $W(S; 10)$. It is easily observed that $10 * R = k+6$ and $(k+6) * T = 10$. Hence, $(k+6) * U = k+6$. If the walk $W(U; k+6)$ is entirely on the bight, then U is balanced and hence $S = RUT$ is balanced, too. It may easily be inferred that $W(U; k+6)$ can have nonempty intersection with the hank only when $n = 17$ and $l(U) = 8$. Indeed, at least four steps of the walk must be on the bight and at least four steps in the hank. Besides, by Lemma 3.1, we cannot have five steps in the hank. This implies $l^+(R) = l(R) = 4 = l(T) = l^-(T)$. Moreover, by inspecting the walk $W(U; 14)$ on the action diagraph Y_{17} we see that U is of the form $U_1 U_2 U_3$, where $l(U_1) = l^+(U_1) = 3 = l^-(U_3) = l(U_3)$. It follows that $S = RUT$ is balanced if and only if U_2 is balanced. That U_2 is balanced can be checked by considering the closed walk $W(S; 8)$.

Subcase 1.2: *The walk W contains the segment* $[8, k+4]$ of B_n.

Here, the argument closely follows the one used in Subcase 1.1. Now we let R be the minimal initial subsequence of S such that $(k+4) * R = 8$ and let T be the minimal terminal subsequence of S such that $8 * T = k+4$. Then S is of the form $S = RUT$, where $8 * U = 8$. As above, $l^+(R) - l^-(R) \geqslant (n-9)/2$ and $l^-(T) - l^+(T) \geqslant (n-9)/2$ and the sequence RT is balanced. Besides, $4 \leqslant l(U) \leqslant 9$. Consequently, $l^-(R), l^+(T) \leqslant 2$. Now we consider the closed walk $W(S; 2k-2)$. We have $(2k-2) * R = k+2$ and $(k+2) * T = 2k-2$. Hence, $(k+2) * U = k+2$. If $W(U; k+2)$ is entirely on the bight then U and hence $S = RUT$ is balanced. Again, we can see that the walk $W(U; k+2)$ can have nonempty intersection with the hank only when $n = 17$ and $l(U) \in \{8, 9\}$. (Indeed, at least four steps of the walk must be on the bight and at least four steps in the hank. Contrary to the previous case, it is possible to use exactly five steps in the hank.) We have $l^-(R) = l(R) = 4 = l(T) = l^+(T)$. By inspecting the walk $W(U; 10)$ on the action digraph Y_{17} we see that U is of the form $U_1 U_2 U_3$, where $l(U_1) = l^-(U_1) = 3 = l^+(U_3) = l(U_3)$. It follows that $S = RUT$ is balanced if and only if U_2 is balanced. That U_2 is balanced can be checked by considering the closed walk $W(S; 16)$.

Case 2: *The walk W is bighty.*

By checking the action digraph Ω_n it is immediately seen that $l(S) \geqslant n-5$. Moreover, S is equivalent to a relation $S' = QR$, where $l^+(S') \geqslant l^-(S')$ and where the walk $W(Q;8)$ covers the bight and has empty intersection with the hank. In other words, $8*Q = 2k$. Clearly, $l^+(Q) \geqslant n-9$. Moreover, by checking the action digraph Ω_n we see that $l^+(R) \geqslant 4$. Therefore, $l^+(S') \geqslant n-5$ and so $l^-(S') \leqslant 5$. Let S'' be the minimal length relation obtained from S' by a (possibly empty) sequence of contractions. (In fact, since any contraction reduces the length by two, at most two contractions are possible.) Observe that since the walk W is bighty, the same holds true for the walk $W(S''; k+4)$. The fact that S'' admits no contraction implies that S'' is equivalent to a relation Z whose code $c(Z)$ (see the definition at the beginning of this section) can have one of the following seven forms:

1. $0^5 1^r$, $r = n-5$;
2. $0^4 1^r$, $n-6 \leqslant r \leqslant n-4$;
3. $0^3 1^r$, $n-7 \leqslant r \leqslant n-3$;
4. $0^2 1^r$, $n-8 \leqslant r \leqslant n-2$;
5. $0^3 1^r 0^2 1^s$, $2 \leqslant r \leqslant s$, $r+s = n-5$;
6. $0^2 1^r 0^2 1^s$, $2 \leqslant r \leqslant s$, $n-6 \leqslant r+s \leqslant n-4$;
7. 1^r, $n-5 \leqslant r \leqslant n$.

We claim that only the form labeled 7 can occur. The first three forms labeled i, $i = 1,2,3$, are excluded by the same argument; we show that the walk $W(Z; 14-i)$, $i = 1,2,3$, is not closed. Indeed, $(14-i)*_{(n-3-i)}Z = 2k$ and, since in order to reach $14-i$ from $2k$ (through the hank) at least $10-i$ steps are needed, we have $n \geqslant l(S) \geqslant n+7-2i$, contradicting the fact that $i \leqslant 3$. The form labeled 4 is excluded next. We may assume that Z is equivalent to $Z' = b^{-1}xy_1y_2T'$, where $x \in \{a^{-1}, b^{-1}\}$ and $y_1, y_2 \in \{a, b\}$. Considering the walks $W(Z'; 2k), W(Z'; 2k-1), \ldots, W(Z'; 8)$ we obtain that T' is either $(ab)^{(n-5)/2}$ or $(ba)^{(n-5)/2}$. Hence, Z' is equivalent to one of the following four sequences:

$$b^{-1}a^{-1}b^2(ba)^{(n-5)/2},$$

$$b^{-1}a^{-1}ba(ba)^{(n-5)/2},$$

$$b^{-2}a^2(ba)^{(n-5)/2},$$

$$b^{-2}ab(ba)^{(n-5)/2}.$$

Since $(ba)^{(n-5)/2} = (5\,6\,7)^{(n-5)}$, a direct computation shows that $3 * Z' = 1$ in the first case, $2 * Z' = 0$ in the second and third case, and $1 * Z' = 3$ in the fourth case. The forms labeled 5 and 6 are again excluded by the same argument. Observe that, with $m = n-2$ for the form labeled 5 and with $m = n-4$ for the form labeled 6, we have $11 *_m Z = 2k$, while $11 *_i Z \in V(B_n) \setminus \{2k\}$ for each $0 \leqslant i < m$. But in at most two (or four) remaining steps we cannot get back to 11.

We conclude that $c(Z) = 1^r$, where $n-5 \leqslant r \leqslant n$. In other words, $n-5 \leqslant l(Z) = l^+(Z) \leqslant n$. To complete the proof of Proposition 3.3, we are now going to show that

Z is equivalent to the relation a^n if $l(Z)$ is odd and to the relation $(ab)^{(n-5)/2}$ if $l(Z)$ is even.

Subcase 2.1: *$l(Z)$ is odd.*

Since $l(Z)$ is odd, $W(Z;k+4)$ must contain a singular arc, forcing $l(S)\in\{n-2,n\}$. Suppose that Z contains a subsequence ab and is therefore equivalent to a relation of the form $P=abT$. Consider the walk $W(P;3)$. Then $3*_2P=8$ and since $l^-(T)=0$ we have that $3*_{n-7}P=2k$. Now the remaining five or seven steps would have to, firstly, go through a singular arc and, secondly, come back to 3. But this is not possible because the edges joining the right hank and the left hank have tails in the right hank (and heads in the left hank). This shows that Z is either a^r or b^r. Since a^n and b^n are indeed relations, it follows that $r\neq n-2$.

Subcase 2.2: *$l(Z)$ is even.*

Observe that Z must be equivalent either to the sequence $P'=xa^2T$ or to the sequence $P''=(ab)^q$, where $q=l(Z)/2$. In the first case consider the walk $W=W(P';5)$. Since $5*_{n-6}P'=2k$, the remaining one, three or five steps would have to take us back to 5. But this is impossible. Hence, the second possibility occurs. In order for the walk $W(P'';8)$ to be closed we see that $l(P'')=n-5$. Moreover, checking the closed walk $W(P'';5)$ we have that $n-5$ is divisible by 3 and so $n\equiv 2 \pmod 3$. The sequence $(ab)^{(n-5)/2}$ is indeed a relation.

Finally, going back to the sequence S, recall that it is equivalent to the relation S'' which is obtained by performing at most two expansions of Z. Note further that expansions preserve the parity of the length of a relation. Hence, if $l(S)$ is odd, then $S'=Z$ and so S is equivalent to a^n. Similarly, for $l(S)$ even we have that S' is obtained by at most two expansions from $(ab)^{(n-5)/2}$ provided $n\equiv 2 \pmod 3$. This concludes the proof of Proposition 3.3. □

4. Proof of Theorem 1.1

We have now gathered all the relevant information on the cycle structure of the graphs X_n. To prove Theorem 1.1 an additional concept is needed. Given a graph X and a 2-path $[u,v,w]$ in X we let $\mathscr{C}(u,v,w)$ denote the set consisting of all possible lengths of cycles containing the 2-path $[u,v,w]$. The following simple observation is an extension of [9], whereas Proposition 4.2 can be easily deduced from [7, Lemma 2.1].

Proposition 4.1. *Let X be a connected graph such that for any two adjacent vertices $u,v\in V(X)$, the sets $\mathscr{C}(u,v,x)$ $(x\in N(v)\setminus\{u\})$ are all distinct. Then no nonidentity automorphism of X fixes two adjacent vertices and furthermore, for each $v\in V(X)$, the group $(\operatorname{Aut}X)_v$ is either trivial or an elementary abelian 2-group.*

Proof. The first part of the statement is observed in [9]. It is also clear that the restriction of each nonidentity automorphism $\alpha\in(\operatorname{Aut}X)_v$ to $N(v)$ must be an involution with both orbits of length 2. Consequently, by the first part, $\alpha^2=1$, and so the stabilizer is either trivial or elementary abelian. $\square$

Proposition 4.2. *Let G be a group and Q a generating set of G such that $1\notin Q=Q^{-1}$. Let $X=\mathrm{Cay}(G,Q)$ and $H=\mathrm{Aut}\,X$. Then $N_H(G)\cap H_1\cong \mathrm{Aut}(G,Q)$.*

If $x\in S_n$ is a permutation of $\wp_n$, we let θ_x denote the action of x on A_n by conjugation, that is $\theta_x(z)=xzx^{-1}$ for each $z\in A_n$.

Lemma 4.3. *Let $n=2k+1\geqslant 17$ and let $a=(0\,1\,\ldots\,2k)\in A_n$ and $b=tat^{-1}$, where $t=(0\,2)(4\,7)\in A_n$. Then $N_{S_n}(a)\cap N_{S_n}(b)$ contains no element x satisfying $\theta_x(a)=a^{-1}$ and $\theta_x(b)=b^{-1}$.*

Proof. Suppose on the contrary that such an element x exists. It follows that $\theta_{txt}(a)=a^{-1}$. A short calculation shows that $x, txt\in\{y_c\colon c\in\wp_n\}$, where $y_c\in S_n$ maps according to the rule $y_c(i)=c-i$ for each $i\in\wp_n$. Hence, $ty_ct=y_d$ for some $c,d\in\wp_n$. Since n is large enough there exists $j\in\wp_n\setminus\{0,2,4,7\}$ such that $c-j\in\wp_n\setminus\{0,2,4,7\}$, too. Consequently, $t(j)=j$ and $t(c-j)=c-j$ and so $c-j=ty_ct(j)=y_d(j)=d-j$, implying $c=d$. Hence, $ty_ct=y_c$. Applying this relation first to $i=0$ and then to $i=4$ we obtain, respectively, $t(c-2)=c$ and $t(c-4)=c-7$. The first condition forces $c=2$, whereas the second one forces $c=11$, a contradiction. $\square$

Proof of Theorem 1.1. Let $G_n=\mathrm{Aut}\,X_n$. It is clear that θ_t interchanges a and b, and so $\theta_t\in\mathrm{Aut}\,(A_n,Q_{a,b})\leqslant(G_n)_1$. Letting $L(A_n)$ denote the left regular representation of A_n, we have $H_n=\langle L(A_n),\theta_t\rangle\leqslant G_n$. Clearly, $L(A_n)$ is a normal subgroup of index 2 in H_n and a short calculation shows that $L(A_n)$ has trivial intersection with $Z(H_n)=\langle\lambda_t\theta_t\rangle\cong Z_2$, where $\lambda_t\in L(A_n)$ is the left translation by the element t. This shows that $H_n\cong A_n\times Z_2$. It remains to show that $H_n=G_n$. We first prove that $[G_n:L(A_n)]$ is either 2 or 4. To this end, let us analyze the structure of the sets $\mathscr{C}(a,1,b)$, $\mathscr{C}(a,1,b^{-1})$ and $\mathscr{C}(a,1,a^{-1})$.

By Proposition 3.2 we have $4\in\mathscr{C}(a,1,b)$, but $4\notin\mathscr{C}(a,1,b^{-1})$ and $4\notin\mathscr{C}(a,1,a^{-1})$. Moreover, by Proposition 3.3(i), we have $n\in\mathscr{C}(a,1,a^{-1})$, but $n\notin\mathscr{C}(a,1,b)$ and $n\notin\mathscr{C}(a,1,b^{-1})$. Therefore, these three sets are pairwise distinct. Using Proposition 4.1 we may deduce that $[G_n:L(A_n)]\in\{2,4\}$ and, more precisely, $(G_n)_1$ is either Z_2 or $Z_2\times Z_2$. Assume that $(G_n)_1\cong Z_2\times Z_2$. Then H_n is of index 2 in G_n and so is normal in G_n. It is easily seen that $L(A_n)$ is a characteristic subgroup of H_n and therefore a normal subgroup of G_n. Hence, $(G_n)_1=N_{G_n}(L(A_n))\cap(G_n)_1$. In view of Proposition 4.2 we have $(G_n)_1=\mathrm{Aut}\,(A_n,Q_{a,b})$. In particular, $\mathrm{Aut}\,(A_n,Q_{a,b})\cong Z_2\times Z_2$. The same holds for its restriction to $Q_{a,b}$. Hence, there exists $\gamma\in\mathrm{Aut}(A_n,Q_{a,b})$ whose restriction to $Q_{a,b}$ is $(a,a^{-1})(b,b^{-1})$. Now recall that, since $n\neq 6$, the automorphisms of A_n are of the form θ_x for $x\in S_n$, and so the existence of such a γ contradicts Lemma 4.3. We conclude that $(G_n)_1\cong Z_2$. So $G_n=H_n=\langle L(A_n),\theta_t\rangle\cong A_n\times Z_2$. In particular, the graph X_n is $\frac{1}{2}$-arc-transitive. $\square$

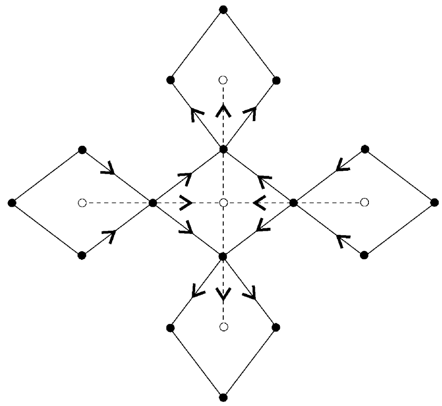

Fig. 3. The orientation of $\mathrm{Alt}(X_n)$.

5. Proof of Theorem 1.2

Recall that $Y_n = \mathrm{Alt}(X_n)$ and that Y_n is indeed a 4-valent graph since the alternating cycles of X_n have length 4. Moreover, $G_n = \mathrm{Aut}\, X_n$ is contained (as an isomorphic copy) in $\mathrm{Aut}\, Y_n$ and acts transitively on the set of vertices and the set of edges of Y_n. The natural orientation of the edge set of X_n gives rise to an orientation of the edge set of Y_n (see Fig. 3), which is preserved by G_n. Hence, G_n acts $\frac{1}{2}$-arc-transitively on Y_n. Further, since the vertex set of Y_n has half the size of that of X_n, we have that the vertex stabilizer of G_n on Y_n has order 4 and is therefore isomorphic to $Z_2 \times Z_2$.

Theorem 1.2 will therefore be proved if we show that this group is the full automorhism group of Y_n. To this end we shall exploit the fact that Y_n has the same automorphism group as its line graph $L(Y_n)$, and set our aim at proving that $L(Y_n)$ and X_n have the same automorphism group.

Observe that $L(Y_n)$ may be obtained from X_n by adding pairs of diagonal edges to each of the alternating cycles. More precisely, $L(Y_n) = \mathrm{Cay}(A_n, \{a, a^{-1}, b, b^{-1}, x, y\}$, where $x = ab^{-1} = (0\,2)(1\,2k)(3\,6)(4\,7) = ba^{-1}$ and $y = a^{-1}b = (0\,2)(1\,3)(4\,7)(5\,8) = b^{-1}a$. To simplify the notation, recall that $Q_{a,b} = \{a, a^{-1}, b, b^{-1}\}$ and let $R_{a,b} = \{x, y\}$. We shall call an edge of $L(Y_n)$ *old* if it is also an edge of X_n and *new* otherwise. In particular, a new edge is an x-*edge* if it is of the form $[v, vx]$ and a y-*edge* if it is of the form $[v, vy]$. By V_n and E_n we shall denote, respectively, the vertex set $V(X_n) = V(L(Y_n))$ and the edge set $E(X_n)$. Moreover, by E_x and E_y we shall denote the sets of all x-edges and y-edges in $L(Y_n)$, respectively. Observe that the group G_n has three edge orbits in $L(Y_n)$, namely the sets E_n, E_x and E_y. We shall see that these three sets are also the edge orbits of $\mathrm{Aut}\, L(Y_n)$, and consequently that the latter coincides with G_n. A feature discriminating between new and old edges is thus needed. We claim that the number of 8-cycles containing a given edge of $L(Y_n)$ will do the trick. Having outlined the strategy of the proof of Theorem 1.2 we now introduce some additional notation.

Let $\mathscr{C}$ be a G_n-orbit of 8-cycles in $L(Y_n)$ and $\mathscr{E}$ an edge G_n-orbit of $L(Y_n)$. We define the *$\mathscr{C}$-frequency* $v_{\mathscr{C}}(\mathscr{E})$ of $\mathscr{E}$ to be the number of cycles in $\mathscr{C}$ containing a given edge $e \in \mathscr{E}$. Similarly, we define the *$\mathscr{E}$-frequency* $\mu_{\mathscr{E}}(\mathscr{C})$ of $\mathscr{C}$ as the number of edges from $\mathscr{E}$ contained on a given cycle in $\mathscr{C}$. We shall use simplified notations $v_{\mathscr{C}} = v_{\mathscr{C}}(E_n)$, $v_{\mathscr{C}}(x) = v_{\mathscr{C}}(E_x)$ and $v_{\mathscr{C}}(y) = v_{\mathscr{C}}(E_y)$. Likewise, $\mu(\mathscr{C}) = \mu_{E_n}(\mathscr{C})$, $\mu_x(\mathscr{C}) = \mu_{E_x}(\mathscr{C})$, $\mu_y(\mathscr{C}) = \mu_{E_y}(\mathscr{C})$. By a straightforward counting argument we obtain the following lemma.

Lemma 5.1. *Let $\mathscr{C}$ be a G_n-orbit of 8-cycles in $L(Y_n)$ and let $\mathscr{E}$ be an edge G_n-orbit of $L(Y_n)$. Then*

$$|\mathscr{E}| v_{\mathscr{C}}(\mathscr{E}) = |\mathscr{C}| \mu_{\mathscr{E}}(\mathscr{C}).$$

Note that an 8-cycle in $L(Y_n)$ arises from an m-cycle in X_n, where $8 \leqslant m \leqslant 16$, by replacing $m-8$ pairs of adjacent terms a and b^{-1} by x or b and a^{-1} by y in the corresponding $Q_{a,b}$-relation of length m in A_n. Cycles of lengths between 8 and 16 in X_n are thus crucial for our analysis. Since by Proposition 3.3 an odd cycle in X_n has length at least n, we only need to check cycles of lengths 8, 10, 12, 14, and 16 in X_n.

Recall that for $t = (0\,2)\,(4\,7)$, the element $\theta_t \in \mathrm{Aut}\,(A_n, Q_{a,b})$ interchanges a and b. A *swap* of a noncontractible $Q_{a,b}$-sequence S in A_n consists in replacing a balanced subsequence of length 2 in S by its image under θ_t. (Note that the first and the last term are also thought of as forming a subsequence of length 2.) Of course, swaps preserve relations (Fig. 4).

Lemma 5.2. *Let $n = 2k+1 \geqslant 17$. Then the only cycles of length at most* 10 *in X_n are the alternating* 4-*cycles.*

Proof. By Proposition 3.3, a cycle of length $l \leqslant 10$ in X_n must necessarily correspond to a balanced $Q_{a,b}$-relation of (even) length $l \leqslant 10$ in A_n, and moreover, to a unique noncontractible $Q_{a,b}$-relation in A_n of (possibly smaller) even length. By Proposition 3.2, the only cycles of length 4 in X_n are the alternating cycles arising from a noncontractible (as well as nonexpandable) relation equivalent to $ab^{-1}ab^{-1}$. Therefore, in order to prove Lemma 5.2, we just need to show that there are no noncontractible relations of lengths 6, 8 or 10. Recalling that swaps preserve relations, a close inspection gives us the following possibilities for the balanced, inequivalent, nonswappable and noncontractible sequences:

(i) *Length* 6:
(1) a^3b^{-3}, (2) $a^3b^{-1}a^{-1}b^{-1}$.

(ii) *Length* 8:
(3) a^4b^{-4}, (4) $a^4b^{-2}a^{-1}b^{-1}$, (5) $a^4b^{-1}a^{-2}b^{-1}$, (6) $a^4b^{-1}a^{-1}b^{-2}$,
(7) $a^2b^2a^{-2}b^{-2}$, (8) $a^2b^{-2}a^2b^{-2}$, (9) $a^2b^2a^{-1}b^{-1}a^{-1}b^{-1}$,

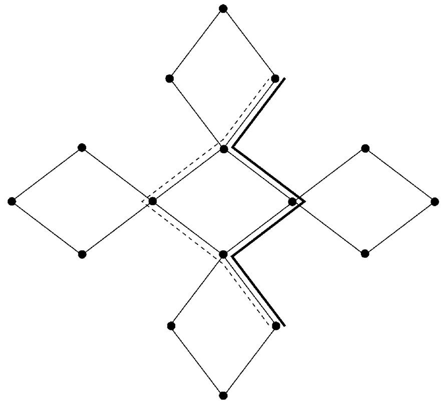

Fig. 4. A swap operation.

(iii) *Length* 10:

(10) a^5b^{-5}, (11) $a^5b^{-3}a^{-1}b^{-1}$, (12) $a^5b^{-1}a^{-3}b^{-1}$, (13) $a^5b^{-1}a^{-1}b^{-3}$,
(14) $a^5b^{-1}a^{-2}b^{-2}$, (15) $a^5b^{-2}a^{-1}b^{-2}$, (16) $a^5b^{-2}a^{-2}b^{-1}$,
(17) $a^5b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$, (18) $a^3b^2a^{-3}b^{-2}$, (19) $a^3b^2a^{-2}b^{-3}$,
(20) $a^3b^2a^{-2}b^{-1}a^{-1}b^{-1}$, (21) $a^3b^2a^{-1}b^{-2}a^{-1}b^{-1}$, (22) $a^3b^2a^{-1}b^{-1}a^{-2}b^{-1}$,
(23) $a^3b^2a^{-1}b^{-1}a^{-1}b^{-2}$, (24) $a^3b^{-3}a^2b^{-2}$, (25) $a^3b^{-2}a^2b^{-3}$,
(26) $a^3b^{-2}a^2b^{-1}a^{-1}b^{-1}$, (27) $a^3b^{-1}a^{-1}b^{-1}a^2b^{-2}$.

Each of these needs to be checked directly against the action digraph Ω_n. It transpires that the vertex 0 is taken to the vertex 2 by sequences numbered (1), (8), (10), (16), (19), (25), and (27); to the vertex $2k-1$ by sequences numbered (2), (11), (15), (18), (20), and (26); to the vertex 3 by sequences numbered (3), (6), (13), (14), and (23); to the vertex 6 by sequences numbered (4), (5), (12), (17), and (22); to the vertex $2k-3$ by sequences numbered (7) and (9); to the vertex 5 by sequences numbered (21) and (24). Thus none of the above is a relation, completing the proof of Lemma 5.2. □

Proof of Theorem 1.2. As mentioned at the beginning of this section, we only need to prove that $\operatorname{Aut} L(Y_n) = G_n$. Clearly, $G_n \leqslant \operatorname{Aut} L(Y_n)$ and moreover, E_n, E_x and E_y are the edge orbits of G_n in $L(Y_n)$. We are now going to prove that $\operatorname{Aut} L(Y_n) \leqslant G_n$. To this end we first show E_n is an edge orbit of $\operatorname{Aut} L(Y_n)$.

Observe that $\operatorname{Aut} L(Y_n)$ cannot have edge orbits E_x and $E_y \cup E_n$, for then the subgraph induced by $E_y \cup E_n$ would have to be edge-transitive, which is not the case since each y-edge lies on two triangles, whereas each old edge lies on a single triangle. Similarly, $\operatorname{Aut} L(Y_n)$ cannot have edge orbits E_y and $E_x \cup E_n$. We conclude that either E_n is an edge orbit of $\operatorname{Aut} L(Y_n)$ or $L(Y_n)$ is edge-transitive. We now prove that the latter cannot occur. Suppose on the contrary that $L(Y_n)$ is edge-transitive. Then the old as well as the new edges have to be contained in the same number of 8-cycles.

In view of Lemma 5.2, an 8-cycle in $L(Y_n)$ arises from a cycle of length 12, 14 or 16 in X_n. For $l=12,14,16$ let v_l denote the number of 8-cycles in $L(Y_n)$ arising from some l-cycle in X_n that contain some fixed old edge of $L(Y_n)$. The numbers v'_l, $l=12,14,16$, have an analogous meaning for the new edges. Let $\mathscr{C}$ be any G_n-orbit of 8-cycles in $L(Y_n)$ arising from a $Q_{a,b}$-relation of length 12 in A_n. Then a cycle from $\mathscr{C}$ contains four old and four new edges, that is $\mu_x(\mathscr{C})+\mu_y(\mathscr{C})=4=\mu(\mathscr{C})$. Note further that $|E_x|=|V_n|/2=|E_y|$ and, of course, $|E_n|=2|V_n)|$. Applying Lemma 5.1 we have $v_{\mathscr{C}}=|\mathscr{C}|/|E_n|\cdot\mu(\mathscr{C})=2|\mathscr{C}|/|V_n|$. On the other hand, $v'_{\mathscr{C}}=|\mathscr{C}|/|E_x|\cdot\mu_x(\mathscr{C})=|\mathscr{C}|/|E_y|\cdot\mu_y(\mathscr{C})=2|\mathscr{C}|/|V_n|\cdot(\mu_x(\mathscr{C})+\mu_y(\mathscr{C}))/2=4|\mathscr{C}|/|V_n|$. It follows that $v'_{12}=2v_{12}$. In a similar way we can obtain $v'_{14}=6v_{14}$. Furthermore, observe that $v_{16}=0$. Namely, an 8-cycle in $L(Y_n)$ arising from a $Q_{a,b}$-relation of length 16 in A_n contains no old edges. Finally, we claim that $v'_{16}=1$. In fact, recall that $(ab^{-1})^2=1$ and observe that $(aba^{-1}b^{-1})^4=1$. Conjugating the latter by a^{-1} results in the relation $(xy)^4=1$. Since every term of a $Q_{a,b}\cup R_{a,b}$-relation of length 8 in A_n arising from a $Q_{a,b}$-relation of length 16 in A_n must be either x or y and since x and y are involutions, $(xy)^4=1$ is the only such relation up to equivalence (with respect to $Q_{a,b}\cup R_{a,b}$). Let $\mathscr{C}$ be the set of all 8-cycles in $L(Y_n)$ arising from the above relation. Clearly, each vertex in $L(Y_n)$ lies on precisely one 8-cycle in $\mathscr{C}$. Therefore $|\mathscr{C}|=|V_n|/8$. Since $\mu_x(\mathscr{C})=4=\mu_y(\mathscr{C})$, it follows by Lemma 5.1 that $v'_{16}=1$.

Using the above information on the frequencies $v_{12},v_{14},v_{16},v'_{12},v'_{14}$ and v'_{16}, we can now see that the number $v_{12}+v_{14}$ of 8-cycles containing an old edge is strictly less than $v'_{12}+v'_{14}+1=2v_{12}+6v_{14}+1$, that is the number of 8-cycles containing a new edge. This contradiction shows that $L(Y_n)$ is not edge-transitive. Hence E_n is an edge orbit of $\operatorname{Aut}L(Y_n)$. Therefore, $\operatorname{Aut}L(Y_n)$ must be contained in the automorphism group of the subgraph induced by the edge orbit E_n. In other words $\operatorname{Aut}L(Y_n)\leqslant G_n$, as required. This completes the proof of Theorem 1.2. □

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